Solution Concepts for Normal-Form Games – Rationalizability

Rationalizability

- Intuitively: strategy is rationalizable if it is a best response to beliefs about strategies of other players
- But it cannot be an arbitrary belief, must take into account rationality

Rationalizability

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Figure 3.6: Matching Pennies game.

- Q. Is playing 'heads' rationalizable?

Rationalizability

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Figure 3.3: The TCP user's (aka the Prisoner's) Dilemma.

- Q. Is playing 'C' rationalizable?

Rationalizability

- Formal definition:
  - For each player *i*, define infinite sequence:

$$S_i^0, S_i^1, S_i^2, \ldots$$

$$S_i^0 = S_i$$

$$S_i^k = \{ s_i : s_i \text{ is best response to some } s_{-i} \in \prod_{j \neq i} CH(S_j^{k-1}) \}$$

Definition 3.4.11 (Rationalizable strategies) *The* rationalizable strategies *for player* i *are* $\bigcap_{k=0}^{\infty} S_i^k$.

Rationalizability

- Nash equilibrium strategies are always rationalizable
- In 2-player games, rationalizable strategies are exactly those strategies that survive iterated removal of strictly dominated strategies.

Solution Concepts for Normal-Form Games – Dominated Strategies

Dominated Strategies

Definition 3.4.8 (Domination) *Let s_i and s'_i be two strategies of player i, and S_{-i} the set of all strategy profiles of the remaining players. Then*

1. s_i strictly dominates s'_i *if for all $s_{-i} \in S_{-i}$, it is the case that* $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

2. s_i weakly dominates s'_i *if for all $s_{-i} \in S_{-i}$, it is the case that* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, *and for at least one $s_{-i} \in S_{-i}$, it is the case that* $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

3. s_i very weakly dominates s'_i *if for all $s_{-i} \in S_{-i}$, it is the case that* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

Dominated Strategies

Definition 3.4.9 (Dominant strategy) *A strategy is* strictly (resp., weakly; very weakly) dominant *for an agent if it strictly (weakly; very weakly) dominates any other strategy for that agent.*

Definition 3.4.10 (Dominated strategy) *A strategy* s_i *is* strictly (weakly; very weakly) dominated *for an agent* i *if some other strategy* s_i' *strictly (weakly; very weakly) dominates* s_i*.*

Dominated Strategies

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

Figure 3.15: A game with dominated strategies.

Dominated Strategies

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

Figure 3.16: The game from Figure 3.15 after removing the dominated strategy R.

Dominated Strategies

	L	C
U	$3, 1$	$0, 1$
D	$0, 1$	$4, 1$

Figure 3.17: The game from Figure 3.16 after removing the dominated strategy M.

Solution Concepts for Normal-Form Games – Minimax regret

Minimax Regret

	L	R
T	$100, a$	$1-\epsilon, b$
B	$2, c$	$1, d$

Minimax Regret

Definition 3.4.5 (Regret) *An agent i's* regret *for playing an action a_i if the other agents adopt action profile a_{-i} is defined as*

$$\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}).$$

Definition 3.4.6 (Max regret) *An agent i's* maximum regret *for playing an action a_i is defined as*

$$\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}) \right).$$

Minimax Regret

Definition 3.4.7 (Minimax regret) *Minimax regret actions for agent i are defined as*

$$\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right].$$

- Q. Why sufficient to look at actions, as opposed to strategies?

Minimax Regret

Definition 3.4.7 (Minimax regret) *Minimax regret actions for agent* i *are defined as*

$$\arg\min_{a_i \in A_i} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a'_i \in A_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right) \right].$$

- Q. Why sufficient to look at actions, as opposed to strategies?

Solution Concepts for Normal-Form Games – Correlated Equilibrium

Correlated Equilibrium

	LW	WL
LW	2, 1	0, 0
WL	0, 0	1, 2

Figure 3.18: Battle of the Sexes game.

- Imagine players condition their results on a coin flip: WL if heads; LW if tails
- Expected payoff: 1.5 for each player

Correlated Equilibrium

Definition 3.4.12 (Correlated equilibrium) *Given an n-agent game $G = (N, A, u)$, a* correlated equilibrium *is a tuple (v, π, σ), where v is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, π is a joint distribution over v, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent i and every mapping $\sigma'_i : D_i \mapsto A_i$ it is the case that*

$$\sum_{d \in D} \pi(d) u_i\left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right) \geq \sum_{d \in D} \pi(d) u_i\left(\sigma_1(d_1), \ldots, \sigma'_i(d_i), \ldots, \sigma_n(d_n)\right).$$

- Mapping is to an action, but allowing mixed strategies adds no greater generality

Correlated Equilibrium

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$$\sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)) \geq \sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma'_i(d_i), \ldots, \sigma_n(d_n)) .$$

- Mapping is to an action, but allowing mixed strategies adds no greater generality
- Every convex combination of C.E.s is a C.E.

Correlated Equilibrium

Theorem 3.4.13 *For every Nash equilibrium* σ^* *there exists a corresponding correlated equilibrium* σ.

Solution Concepts for Normal-Form Games – More Concepts

Trembling-hand perfect eq.

Definition 3.4.14 (Trembling-hand perfect equilibrium) *A mixed-strategy profile s is a* (trembling-hand) perfect equilibrium *of a normal-form game G if there exists a sequence $s^0, s^1, \ldots$ of fully mixed-strategy profiles such that $\lim_{n \to \infty} s^n = s$, and such that for each s^k in the sequence and each player i, the strategy s_i is a best response to the strategies s^k_{-i}.*

- Perfect eq. is stronger than N.E.
- Can require to be robust against small errors (“trembling hand”)

ε-Nash Equilibrium

Definition 3.4.15 (ϵ-Nash) *Fix $\epsilon > 0$. A strategy profile $s = (s_1, \ldots, s_n)$ is an ϵ-Nash equilibrium if, for all agents i and for all strategies $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \epsilon$.*

- Advantages:
  - Always exist
  - Can be computationally useful
- But not necessarily close to a Nash Equilibrium

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- Advantages:
  - Always exist
  - Can be computationally useful
- But not necessarily close to a Nash Equilibrium

ε-Nash Equilibrium

	L	R
U	$1, 1$	$0, 0$
D	$1 + \frac{\epsilon}{2}, 1$	$500, 500$

Figure 3.19: A game with an interesting ϵ-Nash equilibrium.