Computing Solution Concepts – Intro

Solution Concepts

- In single agent settings, there is the notion of *optimal strategy*
- In multiagent setting, situation is more complex. Best strategy depends on the strategies of other agents
- *Solution concepts* – certain subsets of outcomes that are interesting
- Pareto optimality, Nash equilibrium

Computational Concerns

- How to compute a Nash equilibrium?
- Examples we’ve seen had 2 players, each with 2 actions
- Complexity depends on class of games considered
- 2 player zero-sum games
- 2 player general-sum games
- n players, n > 2
- Other solution concepts

Computing Nash Equilibrium in 2-player, zero-sum games

Setup

- Consider 2-player, zero-sum game:

$$G = (\{1, 2\}, A_1 \times A_2, (u_1, u_2))$$

- Let U^*_i be the equilibrium value of player *I*
- Recall in a N.E., player 1's value is equal to his maxmin value

Definition 3.4.1 (Maxmin) *The* maxmin strategy *for player i is* $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, *and the* maxmin value *for player i is* $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

- Use this fact to write an LP

Linear Program

$$
\begin{aligned}
\text{minimize} \quad & U_1^* && && (4.1)\\
\text{subject to} \quad & \sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k \leq U_1^* && \forall j \in A_1 && (4.2)\\
& \sum_{k \in A_2} s_2^k = 1 && && (4.3)\\
& s_2^k \geq 0 && \forall k \in A_2 && (4.4)
\end{aligned}
$$

- Variables: U^*_i, s_2^k

Dual Program

$$\text{maximize} \quad U_1^* \tag{4.5}$$

$$\text{subject to} \quad \sum_{j \in A_1} u_1(a_1^j, a_2^k) \cdot s_1^j \geq U_1^* \qquad \forall k \in A_2 \tag{4.6}$$

$$\sum_{j \in A_1} s_1^j = 1 \tag{4.7}$$

$$s_1^j \geq 0 \qquad \forall j \in A_1 \tag{4.8}$$

Reformulation with Slack Variables

$$\text{minimize} \quad U_1^* \tag{4.9}$$

$$\text{subject to} \quad \sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^* \qquad \forall j \in A_1 \tag{4.10}$$

$$\sum_{k \in A_2} s_2^k = 1 \tag{4.11}$$

$$s_2^k \geq 0 \qquad \forall k \in A_2 \tag{4.12}$$

$$r_1^j \geq 0 \qquad \forall j \in A_1 \tag{4.13}$$

Complexity of Computing a Nash Equilibrium

Computing Nash Equilibria

- 2-player, zero-sum in poly
- 2-player, general sum?
- Cannot be formulated as LP, players not diametrically opposed
- No known reduction from NP-complete problem
- Stumbling block with NP: decision problems. But we always know a NE exists

The PPAD class

- So current knowledge about NE computation is in relation to PPAD class
- PPAD – “Polynomial Parity Argument, Directed Version”
- Family of directed graphs *G(n)*
- Computational task is finding a source or sink node

The family *G(n)*

- Defined on set *N* of 2^n nodes, but described in polynomial space
- Just encode set of edges
- *Parent, Child* functions from *N* to *N:* encoded as arithmetic circuits with sizes poly. in *n*
- An edge exists from node *j* to *k* iff. *Parent(k) = j* and *Child(j) = k*
- There must exist one distinguished node 0 with exactly zero parents
- Find sink or source other than 0 in a given graph

Complexity

Theorem 4.2.1 *The problem of finding a sample Nash equilibrium of a general-sum finite game with two or more players is PPAD-complete.*

- CNE is in PPAD and any other problem in PPAD can be reduced to it
- CNE is in PPAD reduction proceeds quite directly from the proof in the textbook that every game has a NE that uses Sperner's lemma
- Harder part is showing CNE is PPAD-hard. Result was proven in 2005, a culmination of intermediate results achieved over a decade

Complexity

Theorem 4.2.1 *The problem of finding a sample Nash equilibrium of a general-sum finite game with two or more players is PPAD-complete.*

- Not known if P=PPAD. Generally believed not
- It is known that finding an NE in 2 player games is no easier than finding an NE in n player games
- Finding a NE is no easier than finding an arbitrary Brouwer fixed point

LCP Formulation of 2-player NE

Computing Nash Equilibria

- 2-player, zero-sum in poly
- 2-player, general sum?
- Cannot be formulated as LP, players not diametrically opposed
- The LCP formulation

The LCP Formulation

$$\sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^* \qquad \forall j \in A_1 \qquad (4.14)$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k) \cdot s_1^j + r_2^k = U_2^* \qquad \forall k \in A_2 \qquad (4.15)$$

$$\sum_{j \in A_1} s_1^j = 1, \quad \sum_{k \in A_2} s_2^k = 1 \qquad (4.16)$$

$$s_1^j \geq 0, \quad s_2^k \geq 0 \qquad \forall j \in A_1, \forall k \in A_2 \qquad (4.17)$$

$$r_1^j \geq 0, \quad r_2^k \geq 0 \qquad \forall j \in A_1, \forall k \in A_2 \qquad (4.18)$$

$$r_1^j \cdot s_1^j = 0, \quad r_2^k \cdot s_2^k = 0 \qquad \forall j \in A_1, \forall k \in A_2 \qquad (4.19)$$

The Lemke-Howson Algorithm

Lemke-Howson Algorithm

- 2-player, general sum games
- Algorithm is for solving linear complementarity programs
- Searches vertices of strategy simplices (like the simplex algorithm for solving LPs)

Lemke-Howson – a graphical exposition

$0, 1$	$6, 0$
$2, 0$	$5, 2$
$3, 4$	$3, 3$

Figure 4.1: A game for the exposition of the Lemke–Howson algorithm.

Lemke-Howson – a graphical exposition

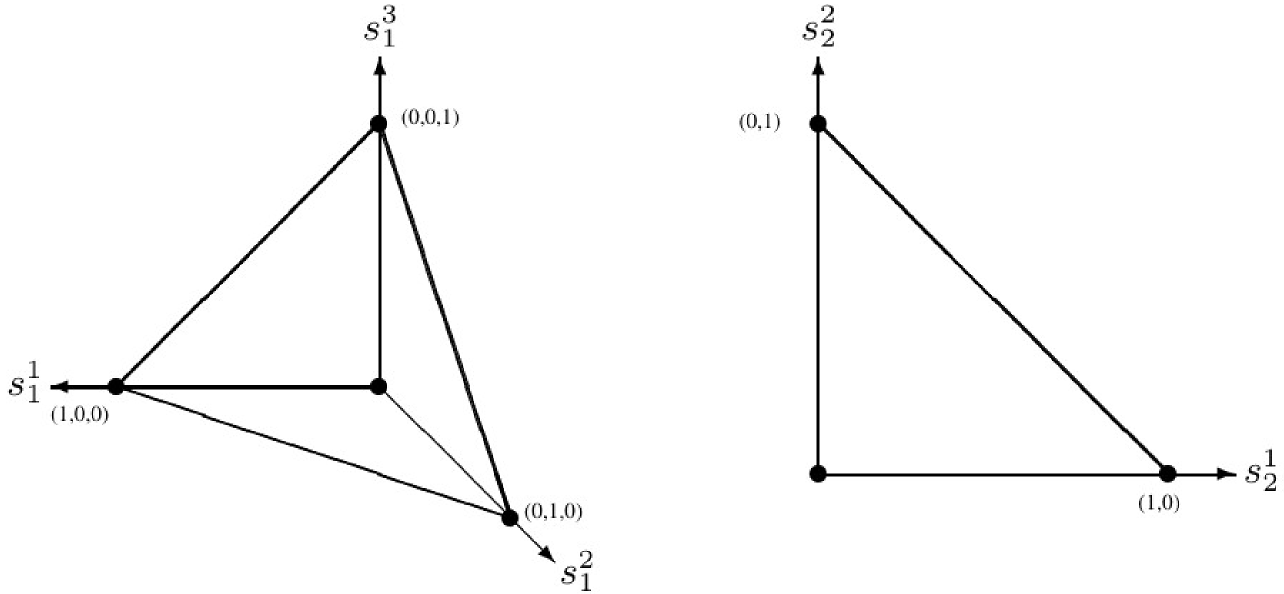

Figure 4.2: Strategy spaces for player 1 (left) and player 2 (right) in the game from Figure 4.1.

Lemke-Howson – a graphical exposition

Our next step in defining the Lemke–Howson algorithm is to define a labeling on the strategies. Every possible mixed strategy s_i is given a set of labels $L(s_i^j) \subseteq A_1 \cup A_2$ drawn from the set of available actions for both players. Denoting a given player as i and the other player as $-i$, mixed strategy s_i for player i is labeled as follows:

- with each of player i's actions a_i^j that is *not* in the support of s_i; and
- with each of player $-i$'s actions a_{-i}^j that *is* a best response by player $-i$ to s_i.

Lemke-Howson – a graphical exposition

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- A strategy profile is a Nash equilibrium iff. it is completely labeled

Lemke-Howson – a graphical exposition

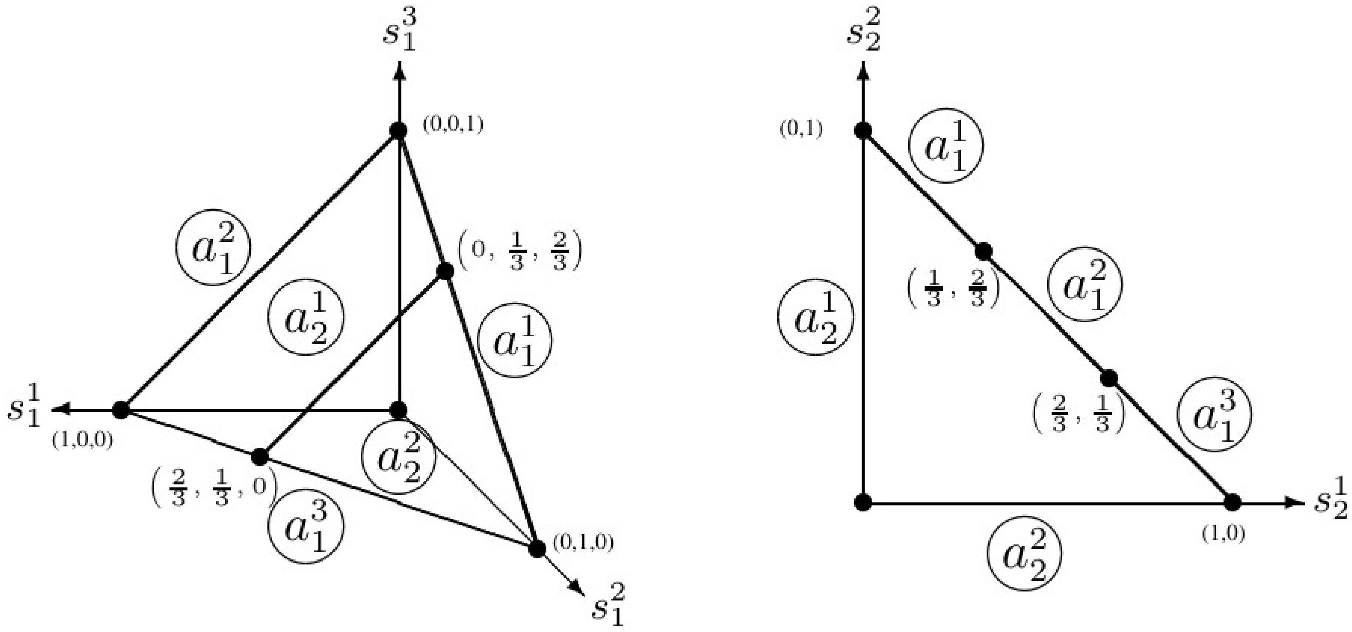
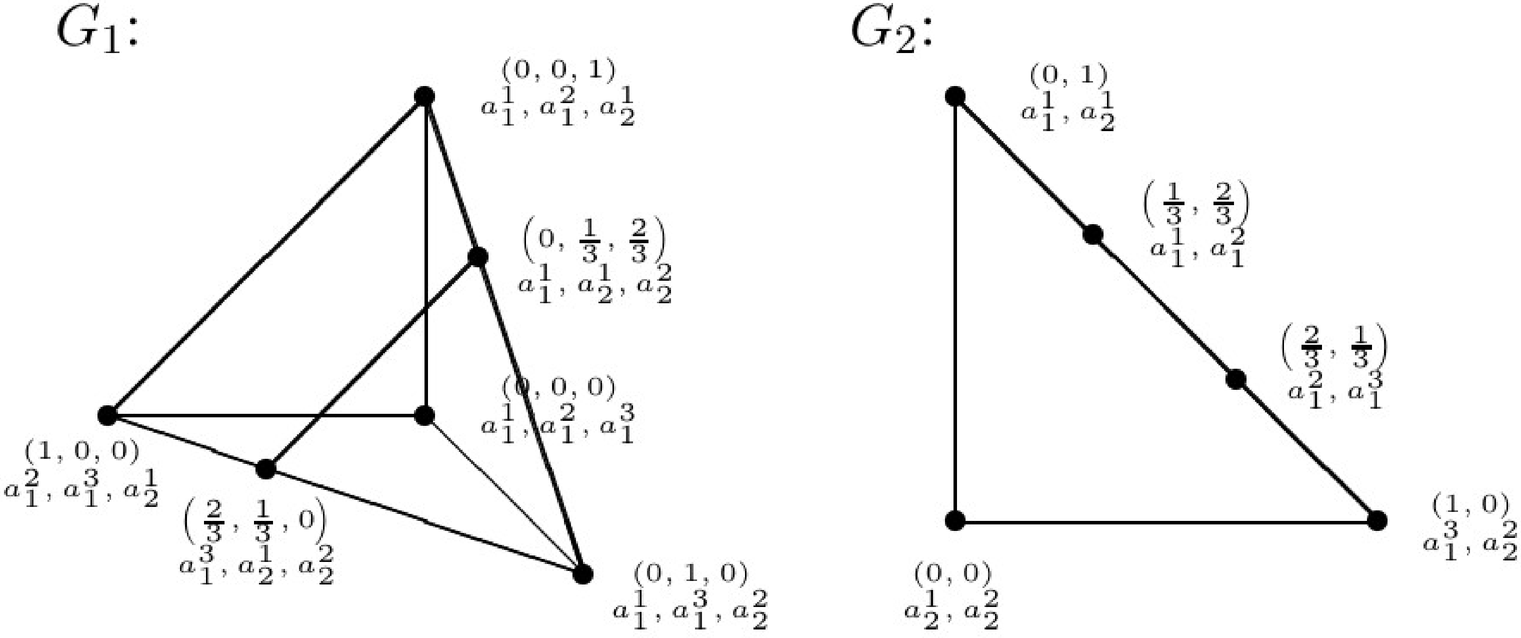


Figure 4.3: Labeled strategy spaces for player 1 (left) and player 2 (right) in the game from Figure 4.1.

Lemke-Howson – a graphical exposition



Lemke-Howson – Properties

- Guaranteed to find a NE
- Alternative proof of the existence of NE
- Path after initial move is unique. Only nondeterminism is in first move
- All paths from the starting point to a NE can be exponential (algorithm is provably exponential)
- No way to assess how close we are to a NE