# Imperfect-Information, Extensive-Form Games

# Imperfect Information

- Up until this point, we’ve assumed players know exactly which node of the game tree they’re in
- Meaning: they have perfect recall of all actions played, including all other players.
- Want to represent two choices made in ignorance of one another

# Imperfect Information

- Partition the nodes into “information sets”, where a player cannot distinguish between two nodes in the same set

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**Definition 5.2.1 (Imperfect-information game)** *An imperfect-information game (in extensive form) is a tuple* $(N, A, H, Z, \chi, \rho, \sigma, u, I)$*, where:*

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ *is a perfect-information extensive-form game; and*
- $I = (I_1, \ldots, I_n)$*, where* $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ *is a set of equivalence classes on (i.e., a partition of)* $\{h \in H : \rho(h) = i\}$ *with the property that* $\chi(h) = \chi(h')$ *and* $\rho(h) = \rho(h')$ *whenever there exists a* $j$ *for which* $h \in I_{i,j}$ *and* $h' \in I_{i,j}$*.*

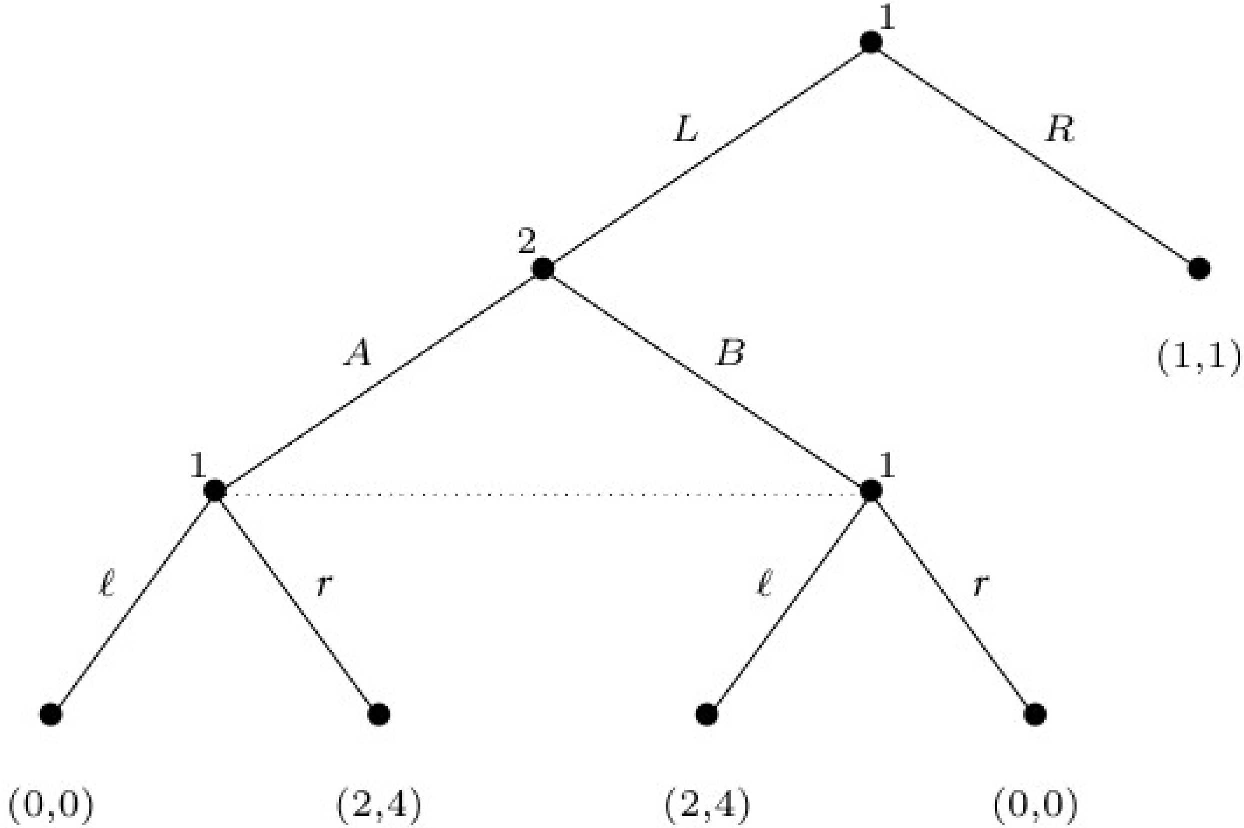


Figure 5.10: An imperfect-information game.

# Imperfect-Information, Extensive-Form Games: Strategies and Equilibria

# Pure Strategies

**Definition 5.2.2 (Pure strategies)** *Let* $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ *be an imperfect-information extensive-form game. Then the pure strategies of player* $i$ *consist of the Cartesian product* $\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$.

# Normal-Form to Extensive-Form

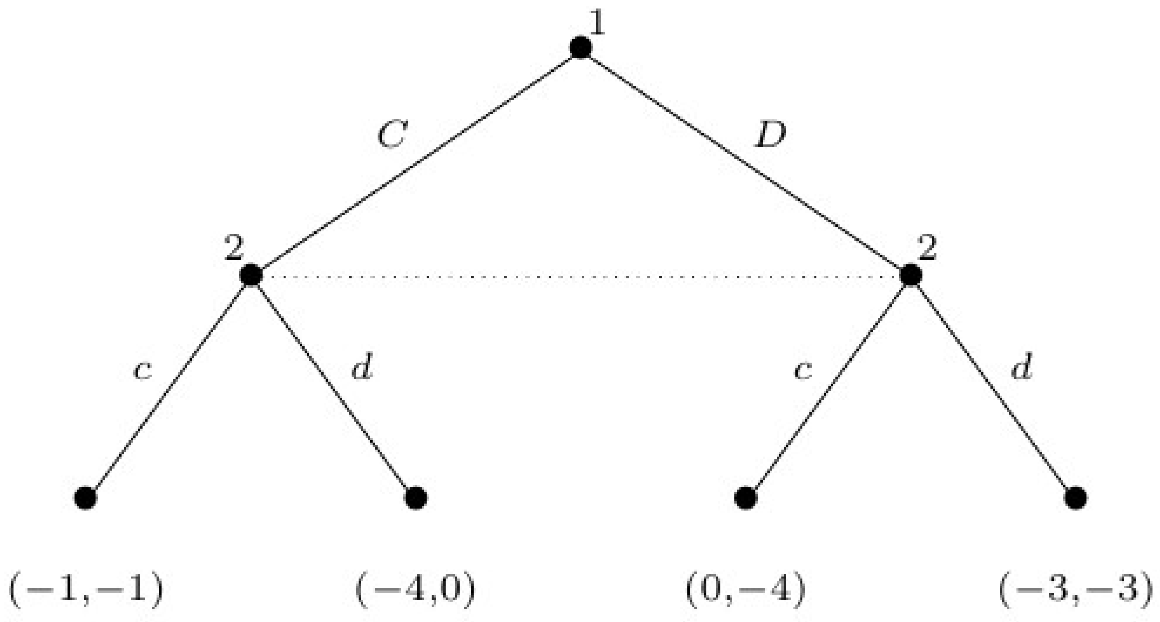


Figure 5.11: The Prisoner's Dilemma game in extensive form.

# Mixed Strategies

- Transform extensive-form game into normal-form (enumerate all pure strategies of each agent)
- Mixed strategy is set of mixed strategies in this image normal-form game
- Can also define *behavioral strategies*
  - Instead of randomizing over pure strategies, randomize independently at each node in a given information set

# Example

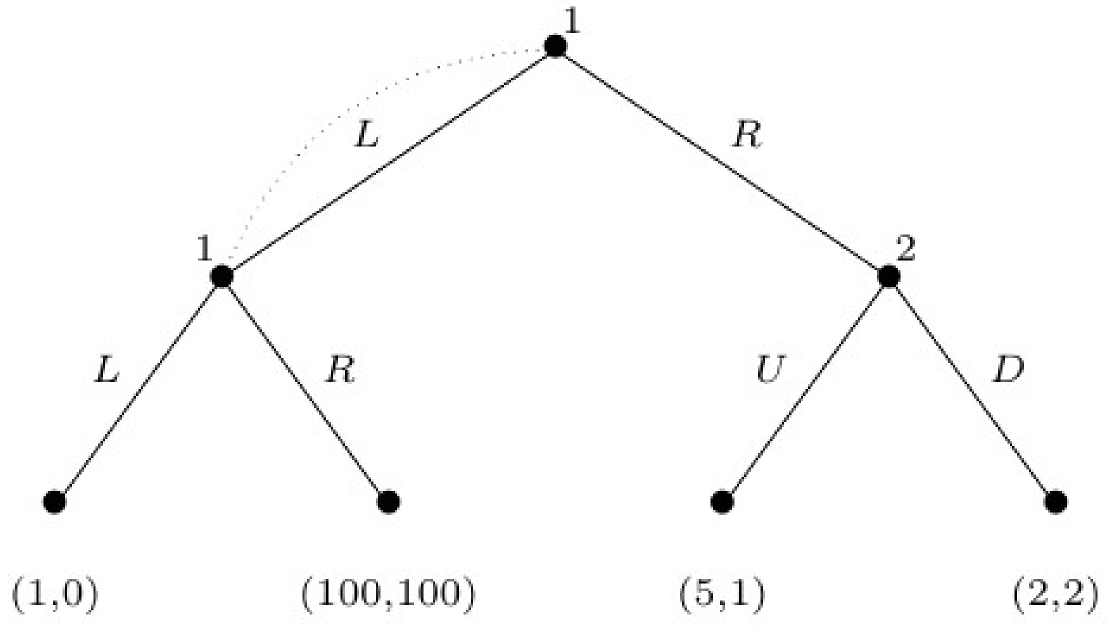


Figure 5.12: A game with imperfect recall

# Perfect Recall

- Every player remembers all of their previous choices

**Definition 5.2.3 (Perfect recall)** *Player $i$ has* perfect recall *in an imperfect-information game $G$ if for any two nodes $h, h'$ that are in the same information set for player $i$, for any path $h_0, a_0, h_1, a_1, h_2, \ldots, h_m, a_m, h$ from the root of the game to $h$ (where the $h_j$ are decision nodes and the $a_j$ are actions) and for any path $h_0, a'_0, h'_1, a'_1, h'_2, \ldots, h'_{m'}, a'_{m'}, h'$ from the root to $h'$ it must be the case that:*

1. $m = m'$;
2. *for all $0 \leq j \leq m$, if $\rho(h_j) = i$ (i.e., $h_j$ is a decision node of player $i$), then $h_j$ and $h'_j$ are in the same equivalence class for $i$; and*
3. *for all $0 \leq j \leq m$, if $\rho(h_j) = i$ (i.e., $h_j$ is a decision node of player $i$), then $a_j = a'_j$.*

*$G$ is a game of perfect recall if every player has perfect recall in it.*

**Theorem 5.2.4 (Kuhn, 1953)** *In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy. Here two strategies are equivalent in the sense that they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining agents.*

# Imperfect-Information, Extensive-Form Games: Sequential Equilibria

# On Equilibria

- Recall that Nash equilibrium was too weak of a concept for perfect information games
- Subgame-perfect equilibrium
- Q: Can we generalize this notion to imperfect information games?

# Subgame-Perfect ?

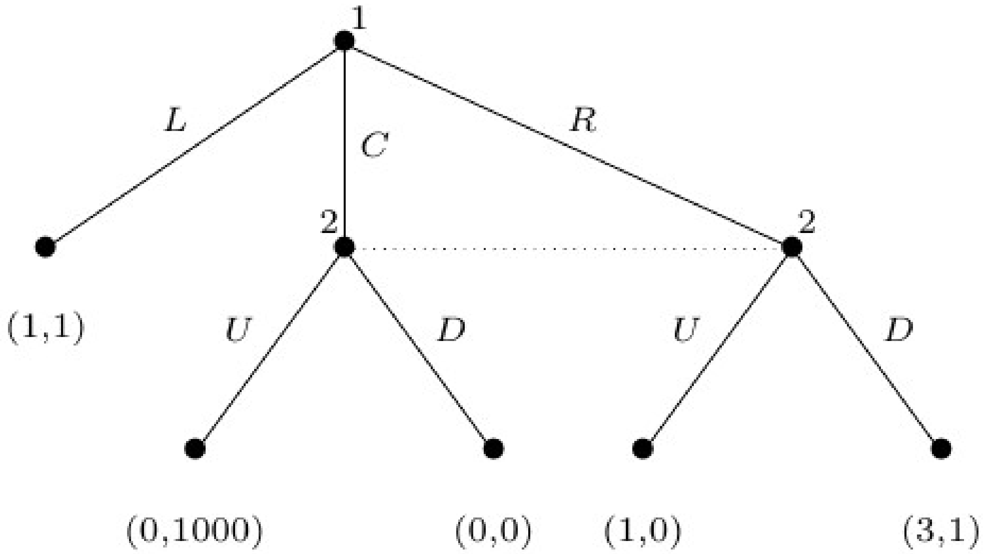


Figure 5.15: Player 2 knows where in the information set he is.

Nash equilibria are $(L, U)$ and $(R, D)$.

# Sequential Equilibrium

- Fully mixed strategy gives positive probability to every node in the game tree
- Makes sense to talk about expected utility given that we are in a specific subtree (using Bayes' rule)
- What about if strategy isn't fully mixed?
- We say probability distribution $\mu(h)$ is consistent with strategy strategy profile if it agrees with Bayes' rule for subtrees with nonzero probability

# Sequential Equilibrium

- Expected utility at *h* given strategy profile *s* and probabiloity distribution:

$$u_i(s \mid h, \mu(h)).$$

**Definition 5.2.10 (Sequential equilibrium)** *A strategy profile $s$ is a* sequential equilibrium *of an extensive-form game $G$ if there exist probability distributions $\mu(h)$ for each information set $h$ in $G$, such that the following two conditions hold:*

1. $(s, \mu) = \lim_{m \to \infty}(s^m, \mu^m)$ *for some sequence* $(s^1, \mu^1), (s^2, \mu^2), \ldots,$ *where $s^m$ is fully mixed, and $\mu^m$ is consistent with $s^m$ (in fact, since $s^m$ is fully mixed, $\mu^m$ is uniquely determined by $s^m$); and*

2. *For any information set $h$ belonging to agent $i$, and any alternative strategy $s'_i$ of $i$, we have that*

$$u_i(s \mid h, \mu(h)) \geq u_i((s', s_{-i}) \mid h, \mu(h)).$$

# Sequential Equilibrium

**Theorem 5.2.11** *Every finite game of perfect recall has a sequential equilibrium.*

**Theorem 5.2.12** *In extensive-form games of perfect information, the sets of subgame-perfect equilibria and sequential equilibria are always equivalent.*

# Imperfect-Information, Extensive-Form Games: Computing Equilibria with Sequence Form

# Sequence-Form Representation

**Definition 5.2.6 (Sequence)** *A* sequence *of actions of player $i \in N$, defined by a node $h \in H \cup Z$ of the game tree, is the ordered set of player $i$'s actions that lie on the path from the root to $h$. Let $\emptyset$ denote the sequence corresponding to the root node. The set of sequences of player $i$ is denoted $\Sigma_i$, and $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ is the set of all sequences.*

**Definition 5.2.7 (Payoff function)** *The* payoff function $g_i : \Sigma \mapsto \mathbb{R}$ *for agent $i$ is given by $g(\sigma) = u(z)$ if a leaf node $z \in Z$ would be reached when each player played his sequence $\sigma_i \in \sigma$, and by $g(\sigma) = 0$ otherwise.*

# Sequence-Form Representation

**Definition 5.2.5 (Sequence-form representation)** *Let $G$ be an imperfect-information game of perfect recall. The* sequence-form representation *of $G$ is a tuple $(N, \Sigma, g, C)$, where*

- *$N$ is a set of agents;*
- *$\Sigma = (\Sigma_1, \ldots, \Sigma_n)$, where $\Sigma_i$ is the set of* sequences *available to agent $i$;*
- *$g = (g_1, \ldots, g_n)$, where $g_i : \Sigma \mapsto \mathbb{R}$ is the payoff function for agent $i$;*
- *$C = (C_1, \ldots, C_n)$, where $C_i$ is a set of linear constraints on the realization probabilities of agent $i$.*

# Example

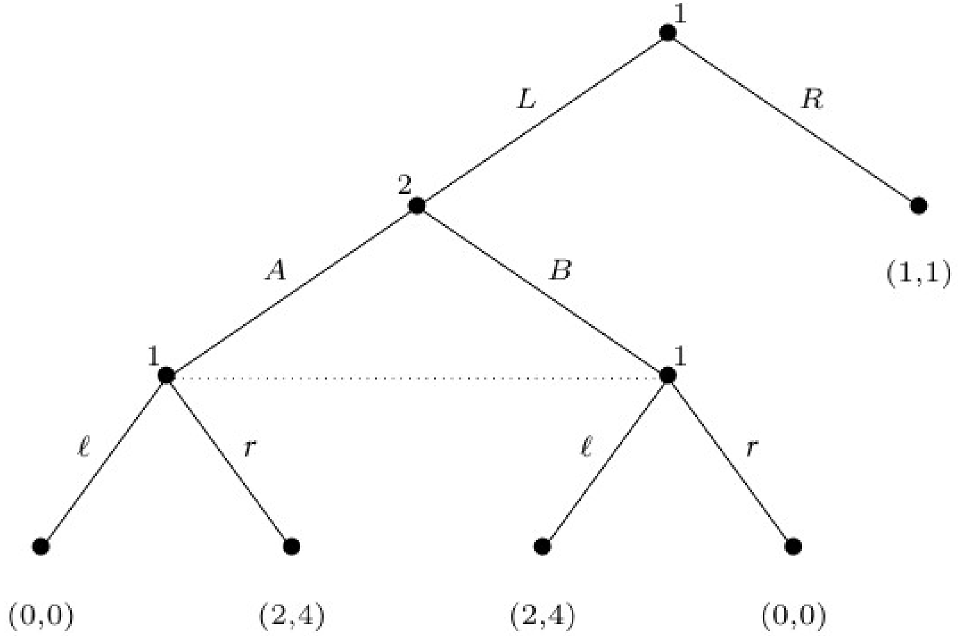


Figure 5.10: An imperfect-information game.

$$\Sigma_1 = \{\emptyset, L, R, L\ell, Lr\} \text{ and } \Sigma_2 = \{\emptyset, A, B\}.$$

# Example

| | $\emptyset$ | $A$ | $B$ |
|---|---|---|---|
| $\emptyset$ | 0, 0 | 0, 0 | 0, 0 |
| $L$ | 0, 0 | 0, 0 | 0, 0 |
| $R$ | 1, 1 | 0, 0 | 0, 0 |
| $L\ell$ | 0, 0 | 0, 0 | 2, 4 |
| $Lr$ | 0, 0 | 2, 4 | 0, 0 |

Figure 5.13: The sequence form of the game from Figure 5.10.

| | $A$ | $B$ |
|---|---|---|
| $L\ell$ | 0, 0 | 2, 4 |
| $Lr$ | 2, 4 | 0, 0 |
| $R\ell$ | 1, 1 | 1, 1 |
| $Rr$ | 1, 1 | 1, 1 |

Figure 5.14: The induced normal form of the game from Figure 5.10.

# Sequence-Form Representation

- But how does player select a sequence? Depending on what other players select, not all sequences are valid
- *behavioral strategy*: probability $\beta_i(h, a_i)$ of taking action $a_i$ at node *h*

**Definition 5.2.8 (Realization plan of $\beta_i$)** *The* realization plan of $\beta_i$ *for player* $i \in N$ *is a mapping* $r_i : \Sigma_i \mapsto [0,1]$ *defined as* $r_i(\sigma_i) = \prod_{c \in \sigma_i} \beta_i(c)$. *Each value* $r_i(\sigma_i)$ *is called a* realization probability.

# Sequence-Form Representation

**Definition 5.2.9 (Realization plan)** *A* realization plan *for player $i \in N$ is a function $r_i : \Sigma_i \mapsto [0,1]$ satisfying the following constraints.*

$$r_i(\emptyset) = 1 \tag{5.1}$$

$$\sum_{\sigma'_i \in \mathrm{Ext}_i(I)} r_i(\sigma'_i) = r_i(\mathrm{seq}_i(I)) \qquad \forall I \in I_i \tag{5.2}$$

$$r_i(\sigma_i) \geq 0 \qquad \forall \sigma_i \in \Sigma_i \tag{5.3}$$

$\beta_i(h, a_i)$ is defined as $\frac{r_i(\mathrm{seq}_i(I)a_i)}{r_i(\mathrm{seq}_i(I))}$

# Computing best response

$$\text{maximize} \quad \sum_{\sigma_1 \in \Sigma_1} \left( \sum_{\sigma_2 \in \Sigma_2} g_1(\sigma_1, \sigma_2) r_2(\sigma_2) \right) r_1(\sigma_1) \qquad (5.4)$$

$$\text{subject to} \quad r_1(\emptyset) = 1 \qquad (5.5)$$

$$\sum_{\sigma_1' \in \mathrm{Ext}_1(I)} r_1(\sigma_1') = r_1(\mathrm{seq}_1(I)) \qquad \forall I \in I_1 \qquad (5.6)$$

$$r_1(\sigma_1) \geq 0 \qquad \forall \sigma_1 \in \Sigma_1 \qquad (5.7)$$

- Dual:

$$\text{minimize} \quad v_0 \qquad (5.8)$$

$$\text{subject to} \quad v_{\mathcal{I}_1(\sigma_1)} - \sum_{I' \in \mathcal{I}_1(\mathrm{Ext}_1(\sigma_1))} v_{I'} \geq \sum_{\sigma_2 \in \Sigma_2} g_1(\sigma_1, \sigma_2) r_2(\sigma_2) \quad \forall \sigma_1 \in \Sigma_1 \qquad (5.9)$$

# Computing equilibria of two-player, zero-sum

$$\text{minimize} \quad v_0 \tag{5.10}$$

$$\text{subject to} \quad v_{\mathcal{I}_1(\sigma_1)} - \sum_{I' \in \mathcal{I}_1(\text{Ext}_1(\sigma_1))} v_{I'} \geq \sum_{\sigma_2 \in \Sigma_2} g_1(\sigma_1, \sigma_2) r_2(\sigma_2) \qquad \forall \sigma_1 \in \Sigma_1 \tag{5.11}$$

$$r_2(\emptyset) = 1 \tag{5.12}$$

$$\sum_{\sigma_2' \in \text{Ext}_2(I)} r_2(\sigma_2') = r_2(\text{seq}_2(I)) \qquad \forall I \in I_2 \tag{5.13}$$

$$r_2(\sigma_2) \geq 0 \qquad \forall \sigma_2 \in \Sigma_2 \tag{5.14}$$