Introduction to Game Theory

Game Theory

- Mathematical study of interaction among independent, self-interested agents
- Applied to economics, political science, biology, psychology, linguistics, and to computer science
- Noncooperative games
- Normal-form games

Self-Interested Agents

- Each agent has her own description of which states or outcomes she likes
- How to model such interests or preferences?
- Utility theory
- Utility function

Example: friends and enemies

- Alice: going to club (*c*), going to a movie (*m*), staying home (*h*)
- By herself, Alice has the utility values of: $u(c) = 100$, $u(m) = 50$, $u(h) = 50$.
- But Alice's utility changes based on other agents: Bob and Carol
- Bob at movies: disutility of 40. Bob at club: disutility of 90.
- Carol: multiply utility by 1.5

Example: friends and enemies

- Bob at movies: disutility of 40. Bob at club: disutility of 90.
- Carol: multiply utility by 1.5
- Bob at club 60% of time, 40% at movies
- Carol at club 25% of time, movies 75% of time

Example: friends and enemies

$A = c$

	$B = c$	$B = m$
$C = c$	15	150
$C = m$	10	100

$A = m$

	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

Figure 3.1: Alice's utility for the actions c and m.

- Expected utility for A=c: 51.75
- Expected utility for A=m: 46.75
- So Alice prefers to go the club even though Bob is often there and Carol rarely is.

Games in normal form

Games

- Under reasonable assumptions about preferences, agents will have utility functions they want to maximize
- Simple if one agent (like in Alice example previously)
- But what if, instead of acting probabilistically, Bob hates Alice and wants to avoid her too?
- And Carol is indifferent to seeing Alice and has a crush on Bob?

Example: TCP user's game

- Imagine you are colleague are only two users on a department network
- Traffic governed by the TCP protocol, which has backoff mechanism when encountering congestion
- Two strategies: Use a correct implementation of TCP (*C*) or defective one (*D*)

Example: TCP user's game

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Figure 3.3: The TCP user's (aka the Prisoner's) Dilemma.

- Given these outcomes, which should you adopt, *C* or *D*?

Example: TCP user's game

- Does your choice depend on what you think your colleague will do?
- From perspective of network operator, what kind of behavior could you expect?
- Under what changes to the delay would the users' decisions still be the same?
- Do answers depend on rationality and / or perception of rationality?

Example: TCP user's game

- Game theory gives answers to many of these questions.
- Any rational user would adopt *D* if playing once or even if playing multiple times
- However, if number of times playing is uncertain or infinite, we may see the users adopt *C*

Definition: Normal-Form Game

- Representation of every player’s utility for every state of the world
- States of the world only depend on players’ combined actions
- More general settings (world depends on randomness in environment as well as players’ actions) can be reduced to normal-form games
- Also normal-form reductions for other game representations

Definition: Normal-Form Game

Definition 3.2.1 (Normal-form game) *A (finite, n-person)* normal-form game *is a tuple (N, A, u), where:*

- N *is a finite set of n players, indexed by i;*
- $A = A_1 \times \cdots \times A_n$*, where A_i is a finite set of* actions *available to player i. Each vector $a = (a_1, \ldots, a_n) \in A$ is called an* action profile*;*
- $u = (u_1, \ldots, u_n)$ *where $u_i : A \mapsto \mathbb{R}$ is a real-valued* utility *(or* payoff*)* function *for player i.*

Example - Prisoner's Dilemma

	C	D
C	a, a	b, c
D	c, b	d, d

Figure 3.4: Any $c > a > d > b$ define an instance of Prisoner's Dilemma.

Example - Common-payoff games

Definition 3.2.2 (Common-payoff game) *A* common-payoff game *is a game in which for all action profiles $a \in A_1 \times \cdots \times A_n$ and any pair of agents i, j, it is the case that $u_i(a) = u_j(a)$.*

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

Figure 3.5: Coordination game.

Example - Zero-Sum Games

Definition 3.2.3 (Constant-sum game) *A two-player normal-form game is* constant-sum *if there exists a constant c such that for each strategy profile $a \in A_1 \times A_2$ it is the case that $u_1(a) + u_2(a) = c$.*

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Figure 3.6: Matching Pennies game.

Example - Coordination and Competition

		Husband	
		LW	WL
Wife	LW	2, 1	0, 0
	WL	0, 0	1, 2

Figure 3.8: Battle of the Sexes game.

Strategies in normal-form games

Strategies

- Recall definition of normal-form game. Each player has a utility function that depends on the action profile selected
- Each player can select an action from her available actions and play it – *pure strategy*
- Each player can also select a probability distribution over some set of their actions and play randomly according to that distribution – *mixed strategy*

Strategies

Definition 3.2.4 (Mixed strategy) *Let* (N, A, u) *be a normal-form game, and for any set* X *let* $\Pi(X)$ *be the set of all probability distributions over* X*. Then the set of* mixed strategies *for player* i *is* $S_i = \Pi(A_i)$.

Definition 3.2.5 (Mixed-strategy profile) *The set of* mixed-strategy profiles *is simply the Cartesian product of the individual mixed-strategy sets,* $S_1 \times \cdots \times S_n$.

Definition 3.2.6 (Support) *The* support *of a mixed strategy* s_i *for a player* i *is the set of pure strategies* $\{a_i | s_i(a_i) > 0\}$.

Definition 3.2.7 (Expected utility of a mixed strategy) *Given a normal-form game* (N, A, u)*, the expected utility* u_i *for player* i *of the mixed-strategy profile* $s = (s_1, \ldots, s_n)$ *is defined as*

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j).$$