Preferences and utility

Preferences and utility

- Why should a real-valued utility function be enough to explain preferences?
- Why should an agent’s response to uncertainty be characterized by the expected value of utility?
- Preferences

Preferences and utility

- Let O be a finite set of outcomes. For any pair $o_1, o_2 \in O$:
  - $o_1 \succeq o_2$
  - $o_1 \sim o_2$
  - $o_1 \succ o_2$
- How do preferences interact with uncertainty?
- Lotteries: $[p_1 : o_1, \ldots, p_k : o_k]$

Axioms of Utility Theory

Axiom 3.1.1 (Completeness) $\forall o_1, o_2,\ o_1 \succ o_2$ *or* $o_2 \succ o_1$ *or* $o_1 \sim o_2$.

Axioms of Utility Theory

Axiom 3.1.1 (Completeness) $\forall o_1, o_2,\ o_1 \succ o_2$ *or* $o_2 \succ o_1$ *or* $o_1 \sim o_2$.

Axiom 3.1.2 (Transitivity) *If* $o_1 \succeq o_2$ *and* $o_2 \succeq o_3$, *then* $o_1 \succeq o_3$.

Axioms of Utility Theory

Axiom 3.1.3 (Substitutability) *If $o_1 \sim o_2$, then for all sequences of one or more outcomes $o_3, \ldots, o_k$ and sets of probabilities $p, p_3, \ldots, p_k$ for which $p + \sum_{i=3}^{k} p_i = 1$, $[p : o_1, p_3 : o_3, \ldots, p_k : o_k] \sim [p : o_2, p_3 : o_3, \ldots, p_k : o_k]$.*

Let $P_\ell(o_i)$ denote the probability that outcome o_i is selected by lottery ℓ. For example, if $\ell = [0.3 : o_1; 0.7 : [0.8 : o_2; 0.2 : o_1]]$, then $P_\ell(o_1) = 0.44$ and $P_\ell(o_3) = 0$.

Axiom 3.1.4 (Decomposability) *If $\forall o_i \in O$, $P_{\ell_1}(o_i) = P_{\ell_2}(o_i)$ then $\ell_1 \sim \ell_2$.*

Axiom 3.1.5 (Monotonicity) *If $o_1 \succ o_2$ and $p > q$ then $[p : o_1, 1 - p : o_2] \succ [q : o_1, 1 - q : o_2]$.*

Axioms of Utility Theory

Lemma 3.1.6 *If a preference relation $\succeq$ satisfies the axioms completeness, transitivity, decomposability, and monotonicity, and if $o_1 \succ o_2$ and $o_2 \succ o_3$, then there exists some probability p such that for all $p' < p$, $o_2 \succ [p' : o_1; (1-p') : o_3]$, and for all $p'' > p$, $[p'' : o_1; (1-p'') : o_3] \succ o_2$.*

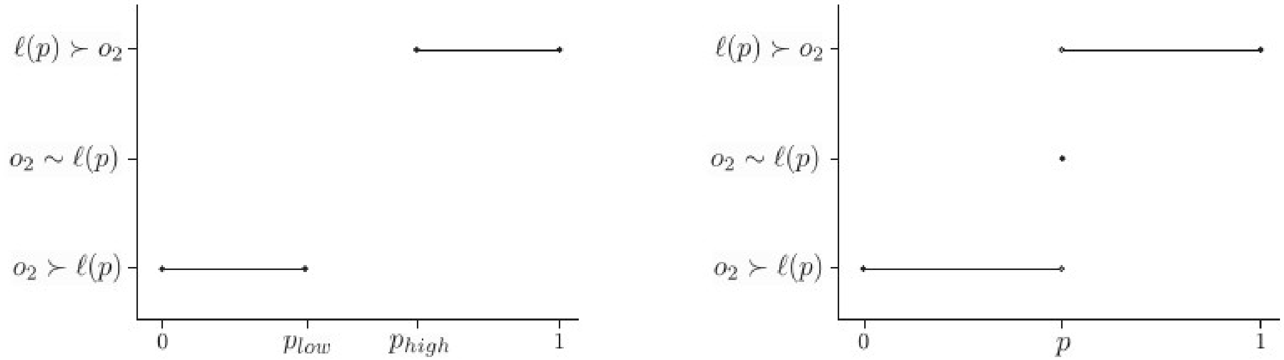

Figure 3.2: Relationship between o_2 and $\ell(p)$.

Axioms of Utility Theory

Proof.

Axioms of Utility Theory

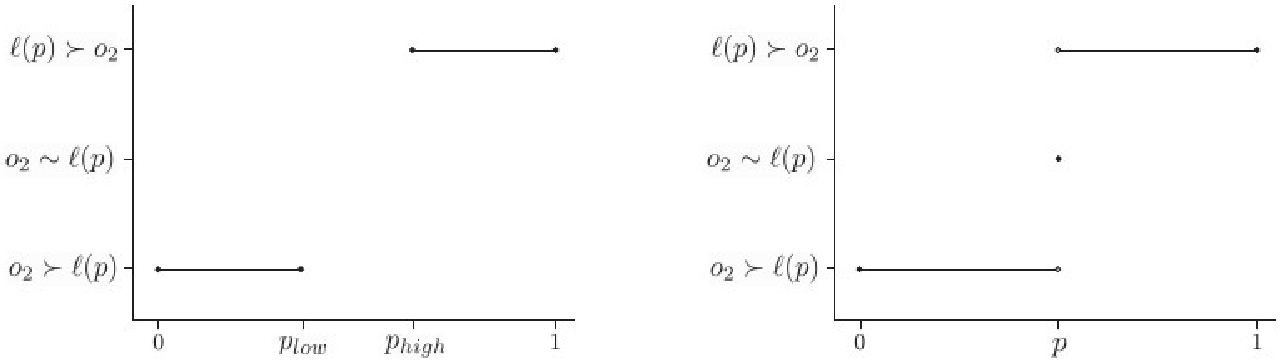

Figure 3.2: Relationship between o_2 and $\ell(p)$.

Axiom 3.1.7 (Continuity) *If* $o_1 \succ o_2$ *and* $o_2 \succ o_3$, *then* $\exists p \in [0, 1]$ *such that* $o_2 \sim [p : o_1, 1 - p : o_3]$.

Characterization

Theorem 3.1.8 (von Neumann and Morgenstern, 1944) *If a preference relation $\succeq$ satisfies the axioms completeness, transitivity, substitutability, decomposability, monotonicity, and continuity, then there exists a function $u : \mathcal{L} \mapsto [0,1]$ with the properties that*

1. $u(o_1) \geq u(o_2)$ *iff* $o_1 \succeq o_2$, *and*
2. $u([p_1 : o_1, \ldots, p_k : o_k]) = \sum_{i=1}^{k} p_i u(o_i)$.