Strategic Value – Normal Form, n-player games, axiomatization

Value of Strategic Game

THEOREM 1. *There is a unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, additivity, and null player. It may be described as follows:*

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^{n} \delta_{i,k}, \tag{3}$$

where $\delta_{i,k}$ denotes the average of $(\delta G)(S)$ over all k-player coalitions S that include i. Furthermore, this map satisfies the axiom of individual rationality.

Axioms of Kohlberg (2021)

- $N = \{1, \ldots, n\}$ is a finite set of players,
- A^i is the finite[16] set of player i's pure strategies, and $A = \prod_{1=1}^{n} A^i$,
- $g^i\colon\ A \to \mathbb{R}$ is player i's payoff function, and $g = (g^i)_{i \in N}$.

We use the same notation, g, to denote the linear extension

- $g^i : \Delta(A) \to \mathbb{R}$,

where for any set K, $\Delta(K)$ denotes the probability distributions on K, and we denote

- $A^S = \prod_{i \in S} A^i$, and
- $X^S = \Delta(A^S)$ (correlated strategies of the players in S).

We define the *direct sum* of strategic games as follows.[17]

Axioms of Kohlberg (2021)

We define the *direct sum* of strategic games as follows.[17]

DEFINITION 1. Let $G_1 = (N, A_1, g_1)$ and $G_2 = (N, A_2, g_2)$ be two strategic games. Then $G := G_1 \oplus G_2$ is the game $G = (N, A, g)$, where $A = A_1 \times A_2$ and $g(a) = g_1(a_1) + g_2(a_2)$.

Denote by $\mathbb{G}(N)$ the set of all n-player strategic games. Let $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$. This may be viewed as a map that associates with any strategic game an allocation of payoffs to the players. We consider a list of axioms for γ. To that end, we first introduce a few definitions.

Let $G \in \mathbb{G}(N)$. We define the *threat power* of coalition S as follows:[19]

$$(\delta G)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \left(\sum_{i \in S} g^i(x, y) - \sum_{i \notin S} g^i(x, y) \right). \tag{2}$$

We say that i and j are *interchangeable* in G if $A^i = A^j$ and $g^i = g^j$; and for any $a, b \in A^N$, if $a^i = b^j$, $a^j = b^i$, and $a^k = b^k$ for all $k \neq i, j$, then $g(a) = g(b)$.

We say that i is a *null player* in G if $g^i(a) = 0$ for all a; and if $a^k = b^k$ for all $k \neq i$, then $g(a) = g(b)$.

Axioms of Kohlberg (2021)

- *Efficiency* $\sum_{i\in N}\gamma_i G = \max_{a\subset A^N}(\sum_{i\in N} g^i(a))$.
- *Balanced threats* If $(\delta G)(S) = 0$ for all $S \subseteq N$, then $\gamma_i = 0$ for all $i \in N$.
- *Symmetry* If i and j are interchangeable in G, then $\gamma_i G = \gamma_j G$.
- *Null player* If i is a null player in G, then $\gamma_i G = 0$.
- *Additivity* $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$.
- *Individual rationality* $\gamma_i(G) \geq \max_{x\in X^i}\min_{y\in X^{N\setminus i}} g^i(x, y)$.

Proof Outline

- Games of threats (like coalitional game for Shapley value)
  - Define Shapley value on GOTs (and is characterized by similar axioms)
- Mapping δ takes game to GOT.
- Characterize δ on some simple games (but complex enough to be onto the class of GOTs) → uniqueness
- Show δ satisfies all the axioms

Games of threats

A *coalitional game of threats* is a pair (N, d), where

- $N = \{1, \ldots, n\}$ is a finite set of players.
- $d\colon 2^N \to \mathbb{R}$ is a function such that $d(S) = -d(N \backslash S)$ for all $S \subseteq N$.

Let $\psi\colon \mathbb{D}(N) \to \mathbb{R}^n$. This may be viewed as a map that associates with any game of threats an allocation of payoffs to the players. Following Shapley (1953), we consider the following axioms.

For all games of threats (N, d), (N, d_1), (N, d_2), and for all players i, j,

- *Efficiency* $\sum_{i \in N} \psi_i d = d(N)$.
- *Symmetry* $\psi_i d = \psi_j d$ if i and j are interchangeable in d (i.e., if $d(S \cup i) = d(S \cup j)$ $\forall S \subseteq N \backslash \{i, j\}$).
- *Null player* $\psi_i d = 0$ if i is a null player in d (i.e., if $d(S \cup i) = d(S)$ $\forall S \subseteq N$).
- *Additivity* $\psi(d_1 + d_2) = \psi d_1 + \psi d_2$.

Games of threats

PROPOSITION 4. *There exists a unique map $\psi : \mathbb{D}(N) \to \mathbb{R}^n$ satisfying the axioms of efficiency, symmetry, null player, and additivity. It may be described as follows:*

$$\psi_i d = \frac{1}{n} \sum_{k=1}^{n} d_{i,k}, \tag{8}$$

where $d_{i,k}$ denotes the average of $d(S)$ over all k-player coalitions that include i.

DEFINITION 2. Let $T \subseteq N$, $T \neq \varnothing$. The unanimity game of threats, $u_T \in \mathbb{D}(N)$, is defined by

$$u_T(S) = \begin{cases} |T| & \text{if } S \supseteq T, \\ -|T| & \text{if } S \subseteq N \setminus T, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5. *Every game of threats is a linear combination of the unanimity games of threats u_T.*

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Mapping δ takes game to GOT

In this section, we present properties of the mapping $\delta\colon \mathbb{G}(N) \to \mathbb{D}(N)$ that are needed for the proof of the main result.

Let $G \in \mathbb{G}(N)$. For any $S \subseteq N$, let $(\delta G)(S)$ be as in (2).

Lemma 1. *δG is a game of threats.*

Proof. By the minmax theorem, $(\delta G)(S) = -(\delta G)(N \setminus S)$ for any $S \subseteq N$. □

We refer to δG as the game of threats associated with G.

Lemma 2. *$\delta : \mathbb{G}(N) \to \mathbb{D}(N)$ satisfies:*

- $\delta(G_1 \oplus G_2) = \delta G_1 + \delta G_2$ *for any* $G_1, G_2 \in \mathbb{G}(N)$.
- $\delta(\alpha G) = \alpha \delta G$ *for any* $G \in \mathbb{G}(N)$ *and* $\alpha \geq 0$.

Mapping δ takes game to GOT

PROOF. Let val(G) denote the minmax value of the two-person zero-sum strategic game G. Then $\text{val}(G_1 \oplus G_2) = \text{val}(G_1) + \text{val}(G_2)$.

To see this, note that by playing an optimal strategy in G_1 as well as an optimal strategy in G_2, each player guarantees the payoff $\text{val}(G_1) + \text{val}(G_2)$.

Now apply the above to all two-person zero-sum games played between a coalition S and its complement $N \setminus S$, as indicated in (2). □

The next lemma is an immediate consequence of the definition of δ.

LEMMA 3. $\delta : \mathbb{G}(\mathbb{N}) \to \mathbb{D}(\mathbb{N})$ *satisfies:*

- $(\delta G)(N) = \max_{a \in A^N}(\sum_{i \in N} g^i(a))$.
- *If i and j are interchangeable in G then i and j are interchangeable in δG.*
- *If i is a null player in G, then i is a null player in δG.*

Proof Outline

- Games of threats (like coalitional game for Shapley value)
  - Define Shapley value on GOTs (and is characterized by similar axioms)
- Mapping δ takes game to GOT.
- **Characterize δ on some simple games (but complex enough to be onto the class of GOTs) → uniqueness**
- Show δ satisfies all the axioms

Characterize on simple games

DEFINITION 3. Let $T \subseteq N$, $T \neq \varnothing$. The unanimity strategic game on T, henceforth the unanimity game on T, is $U_T = (N, A, g_T)$, where

$A^i = \{0, 1\}$ for all $i \in N$,

$g_T(a) = 1_T$ if $a^i = 1$ for all $i \in T$, and $g_T(a) = 0$ otherwise.

That is, if all the members of T consent then they each receive 1; however, if even one member dissents, then all receive zero; the players outside T always receive zero.

LEMMA 4. *Let $T \neq \varnothing$, and let $U_T \in \mathbb{G}(N)$ be the unanimity game on T and $u_T \in \mathbb{D}(N)$ be the unanimity game of threats on T. Then $\delta U_T = u_T$.*

PROOF. Consider the two-person zero-sum game between S and $N \setminus S$.

If $S \cap T$ is neither $\varnothing$ nor T, then both S and $N \setminus S$ include a player in T. If these players dissent, then all players receive 0. Thus, the minmax value, $(\delta U_T)(S)$, is 0.

If $S \cap T = T$ then, by consenting, the players in S can guarantee a payoff of 1 to each player in T and 0 to all the others. Thus, $(\delta U_T)(S) = |T|$.

If $S \cap T = \varnothing$ then, by consenting, the players in $N \setminus S$ can guarantee a payoff of 1 to each player in $T \subset N \setminus S$ and 0 to all the others. Thus, $(\delta U_T)(S) = -|T|$.

By Definition 2, $\delta U_T = u_T$. $\square$

Characterize on simple games

Definition 4. The antiunanimity game on T is $V_T = (N, A, g)$, where $A^i = \{S \subseteq T : S \neq \emptyset\}$ and $g(S_1, \ldots, S_n) = \sum_{i \in T} -1_{S_i}$.

That is, each player in T chooses a nonempty subset of T where each member loses 1. Players outside T also choose such subsets, but their choices have no impact. Thus, the payoff to any player, i, is minus the number of players in T whose chosen set includes i.

Lemma 5. $\delta V_T = -u_T$.

Proof. Let S be a subset of N such that $T \subseteq S$. In the zero-sum game between S and its complement, each player in S chooses a subset of T of size 1. Thus, $(\delta V_T)(S) = -|T|$.

Let S be a subset of N such that $T \cap S \neq \emptyset$ and $T \setminus S \neq \emptyset$. In the zero-sum game between S and its complement, the minmax strategies are for the players in S to choose $T \setminus S$ and for the players in $N \setminus S$ to choose $T \cap S$. The resulting payoff is $-t_1 t_2 - (-t_2 t_1) = 0$, where t_1 and t_2 are the number of elements of $T \cap S$ and $T \setminus S$, respectively. Thus, $(\delta V_T)(S) = 0$.

Therefore, $\delta V_T = -u_T$. ◻

Lemma 6. *For every game of threats $d \in \mathbb{D}(N)$, there exists a strategic game $U \in \mathbb{G}(N)$ such that $\delta U = d$. Moreover, there exists such a game that can be expressed as a direct sum of nonnegative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the antiunanimity games $\{V_T\}_{T \subseteq N}$.*

Proof. By Proposition 5, d is a linear combination of the unanimity games of threats u_T.

$$d = \sum_T \alpha_T u_T - \sum_T \beta_T u_T \quad \text{where } \alpha_T, \beta_T \geq 0 \text{ for all } T.$$

By Lemmas 4 and 5,

$$d = \sum_T \delta(\alpha_T U_T) + \sum_T \delta(\beta_T V_T),$$

and, by Lemma 2,

$$d = \delta\left(\left(\bigoplus_{T \subseteq N} \alpha_T U_T\right) \oplus \left(\bigoplus_{T \subseteq N} \beta_T V_T\right)\right),$$

where $\ominus_T$ stands for the direct sum of the games parameterized by T. ∎

Remark 14. In particular, Lemma 6 establishes that the mapping $\delta\colon \mathbb{G}(N) \to \mathbb{D}(N)$ is onto.

Characterize on simple games

Lemma 7. *For every $G \in \mathbb{G}(N)$, there exists a δ-inverse, that is, $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0$. Moreover, if $G' \in \mathbb{G}(N)$ is such that $\delta G' = \delta G$ then there exists $U \in \mathbb{G}(N)$ that is a $\delta-$inverse of both G and G'.*

Proof. Consider $-\delta G \in \mathbb{D}(N)$. By Lemma 6, there exists $U \in \mathbb{G}(N)$ such that $-\delta G = \delta U$. By Lemma 2, $\delta(G \oplus U) = 0$. And if G' is such that $\delta G' = \delta G$ then, by the same argument, $\delta(G' \ominus U) = 0$. ∎

Proposition 7. *If $\gamma : \mathbb{G}(N) \to \mathbb{R}^n$ satisfies the axioms of balanced threats, efficiency, and additivity, then γG is a function of δG.*

Proof. Let $G, G' \in \mathbb{G}(N)$ be such that $\delta G = \delta G'$. We must show that $\gamma G = \gamma G'$. By Lemma 7, there exists $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0 = \delta(G' \oplus U)$. By the axiom of balanced threats, $\gamma(G \oplus U) = 0 = \gamma(G' \oplus U)$. Thus, by the additivity axiom, $\gamma G = -\gamma U = \gamma G'$. ∎

Characterize on simple games

Lemma 8. *For any $T \neq \emptyset$ and $\alpha \geq 0$, the axioms of symmetry, null player, and efficiency determine γ on the game αU_T. Specifically, $\gamma(\alpha U_T) = \alpha 1_T$.*

Proof. Any $i \notin T$ is a null player in U_T, and so $\gamma_i = 0$. Any $i, j \in T$ are interchangeable in U_T, and so $\gamma_i = \gamma_j$. By efficiency, the sum of the γ_i is the maximum total payoff, which since $\alpha > 0$, is $\alpha|T|$. Thus, each of the $|T|$ nonzero γ_i is equal to α. □

Lemma 9. *For any $\alpha \geq 0$, the axioms (of symmetry, null player, additivity, balanced threats, and efficiency) determine γ on the game αV_T. Specifically, $\gamma(\alpha V_T) = -\alpha 1_T$.*

Proof. By Lemma 8, the axioms determine $\gamma(\alpha U_T) = \alpha 1_T$. By Lemmas 4 and 5, $\delta(\alpha V_T \oplus \alpha U_T) = 0$. Therefore, by the axiom of balanced threats, $\gamma(\alpha V_T \oplus \alpha U_N) = 0$. Thus, by additivity, $\gamma(\alpha V_T) = -\gamma(\alpha U_T) = -\alpha 1_T$. □

Finish proof

PROPOSITION 8. *The map γ of formula (3) satisfies the axiom of individual rationality.*

PROOF OF THEOREM 1. We first prove uniqueness. Let $G \in \mathbb{G}(N)$. Consider $\delta G \in \mathbb{D}(N)$; by Lemma 6 there exists a game $U \in \mathbb{G}(N)$ that is a direct sum of nonnegative multiples of the unanimity games $\{U_T\}_{T\subseteq N}$ and the antiunanimity games $\{V_T\}_{T\subseteq N}$, such that $\delta G = \delta U$.

By Proposition 7, $\gamma G = \gamma U$ and so it suffices to show that γU is determined by the axioms.

Now, by Lemmas 8 and 9, γ is determined on nonnegative multiples of the unanimity games $\{U_T\}_{T\subseteq N}$ and the antiunanimity games $\{V_T\}_{T\subseteq N}$. It then follows from the axiom of additivity that γ is determined on U.

To prove existence, we show that the value, $\gamma = \psi \circ \delta$, satisfies the axioms.

Efficiency, symmetry, and the null player axiom follow from Lemma 3 and the corresponding properties of the Shapley value ψ.

Additivity follows from Lemma 2 and the linearity of the Shapley value.

The axiom of balanced threats follows from formula (3). If $(\delta G)(S) = 0$ for all $S \subseteq N$, then $\gamma_i G = 0$ for all $i \in N$.

Finally, Proposition 8 establishes that γ satisfies the axiom of individual rationality. □