Solution Concepts for Normal-Form Games – Intro

Solution Concepts

- In single agent settings, there is the notion of *optimal strategy*
- In multiagent setting, situation is more complex. Best strategy depends on the strategies of other agents
- *Solution concepts* – certain subsets of outcomes that are interesting
- Pareto optimality, Nash equilibrium

Pareto Optimality

- For an outside observer, can some outcomes of a game be said to be better than another?
- Sum of agents utilities? Except utility functions only encode preferences, not directly comparable between agents
- However, if all agents' utilities increase, that is clearly better

Pareto Optimality

Definition 3.3.1 (Pareto domination) *Strategy profile* s Pareto dominates *strategy profile* s' *if for all* $i \in N$, $u_i(s) \geq u_i(s')$, *and there exists some* $j \in N$ *for which* $u_j(s) > u_j(s')$.

Definition 3.3.2 (Pareto optimality) *Strategy profile* s *is* Pareto optimal, *or* strictly Pareto efficient, *if there does not exist another strategy profile* $s' \in S$ *that Pareto dominates* s.

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Figure 3.3: The TCP user's (aka the Prisoner's) Dilemma.

Pareto Optimality

- Every game must have at least one Pareto optimum. And must have one with pure strategies
- Some games will have multiple – in zero-sum games, all strategy profiles are Pareto optimal
- In common-payoff games, all Pareto optimal strategy profiles have the same payoffs

Solution Concepts for Normal-Form Games – Nash Equilibrium

Nash Equilibrium

- Suppose a player knew how the other players were going to play
- Then the optimal strategy is simple, like single agent setting

Definition 3.3.3 (Best response) *Player i's* best response *to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$.*

Nash Equilibrium

Definition 3.3.4 (Nash equilibrium) *A strategy profile* $s = (s_1, \ldots, s_n)$ *is a* Nash equilibrium *if, for all agents* i*,* s_i *is a best response to* s_{-i}*.*

Definition 3.3.5 (Strict Nash) *A strategy profile* $s = (s_1, \ldots, s_n)$ *is a* strict Nash equilibrium *if, for all agents* i *and for all strategies* $s'_i \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition 3.3.6 (Weak Nash) *A strategy profile* $s = (s_1, \ldots, s_n)$ *is a* weak Nash equilibrium *if, for all agents* i *and for all strategies* $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq$

Examples

	LW	WL
LW	2, 1	0, 0
WL	0, 0	1, 2

Figure 3.9: Pure-strategy Nash equilibria in the Battle of the Sexes game.

- Battle of the Sexes also has a mixed strategy equilibrium:

Examples

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Figure 3.10: The Matching Pennies game.

How to interpret mixed strategies?

Consider penalty kick in soccer game

- Possible interpretations:
  - Roll dice in head
  - Each player's assessment of how likely the other players are to deterministically select an action
  - Game played many times repeatedly
  - Pure-strategy agents selected randomly from larger pool

Solution Concepts for Normal-Form Games – Existence of Nash Equilibrium

Brouwer Fixed Point Theorem

Definition 3.3.7 (Convexity) *A set* $C \subset \mathbb{R}^m$ *is* convex *if for every* $x, y \in C$ *and* $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$. *For vectors* $x^0, \ldots, x^n$ *and nonnegative scalars* $\lambda_0, \ldots, \lambda_n$ *satisfying* $\sum_{i=0}^{n} \lambda_i = 1$, *the vector* $\sum_{i=0}^{n} \lambda_i x^i$ *is called a* convex combination *of* $x^0, \ldots, x^n$.

Definition 3.3.15 (Compactness) *A subset of* $\mathbb{R}^n$ *is* compact *if the set is closed and bounded.*

- Let *K* be a convex, compact set of Euclidean space. Then any continuous function from *K* to *K* has a fixed point.

Existence of Nash equilibria

Theorem 3.3.22 (Nash, 1951) *Every game with a finite number of players and action profiles has at least one Nash equilibrium.*

Proof. Given a strategy profile $s \in S$, for all $i \in N$ and $a_i \in A_i$ we define

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i, s_{-i}) - u_i(s)\}.$$

We then define the function $f : S \mapsto S$ by $f(s) = s'$, where

$$\begin{aligned} s'_i(a_i) &= \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} s_i(b_i) + \varphi_{i,b_i}(s)} \\ &= \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}. \end{aligned} \qquad (3.5)$$

Existence of Nash equilibria

Theorem 3.3.22 (Nash, 1951) *Every game with a finite number of players and action profiles has at least one Nash equilibrium.*

Solution Concepts for Normal-Form Games – Maxmin

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Maxmin Value

Definition 3.4.1 (Maxmin) *The* maxmin strategy *for player i is* $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, *and the* maxmin value *for player i is* $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

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Husband

Wife		LW	WL
	LW	$2, 1$	$0, 0$
	WL	$0, 0$	$1, 2$

Figure 3.8: Battle of the Sexes game.

Minmax Value

Definition 3.4.2 (Minmax, two-player) *In a two-player game, the* minmax strategy *for player i against player $-i$ is* $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, *and player $-i$'s* minmax value *is* $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Definition 3.4.3 (Minmax, n-player) *In an n-player game, the* minmax strategy *for player i against player $j \neq i$ is i's component of the mixed-strategy profile s_{-j} in the expression* $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$, *where $-j$ denotes the set of players other than j. As before, the* minmax value *for player j is* $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$.

Minimax Theorem

Theorem 3.4.4 (Minimax theorem (von Neumann, 1928)) *In any finite, two-player, zero-sum game, in any Nash equilibrium*[5] *each player receives a payoff that is equal to both his maxmin value and his minmax value.*