1 (a) If alg. makes mistake at least half of weight is discounted by $(1-\eta)$

$$\therefore \Phi^{(t+1)} = \sum_{\substack{i \text{ correct} \\ \text{at round } t}} w_i^{(t)} + \sum_{\substack{i \text{ incorrect} \\ \text{at round } t}} w_i^{(t)}$$

$$= \quad '' \quad + (1-\eta) \sum_{i \text{ incorrect}} w_i(t)$$

$$= \Phi^{(t)} - \eta \cdot \sum_{i \text{ incorrect}} w_i(t)$$

$$\leq \Phi^{(t)} - \frac{\eta}{2} \Phi^{(t)} = \left(1 - \frac{\eta}{2}\right) \Phi^{(t)}$$

+3

(b) Initially, $\Phi^{(0)} = n$. Each mistake decreases by at least $\left(1-\frac{\eta}{2}\right)$ factor by (a).

$$\therefore \Phi^{(t)} \leq n \cdot \left(1 - \frac{\eta}{2}\right)^{M(T)}$$

+2

(c) By (b)

$$n \cdot \left(1-\frac{\eta}{2}\right)^{M(T)} \geq \Phi^{(T+1)} \geq w_i^{(T+1)} = (1-\eta)^{m_i^{(T)}}$$

$$\Rightarrow \log(n) + M(T) \log\left(1 - \frac{\eta}{2}\right) \geq m_i^{(T)} \log(1-\eta)$$

$$\Rightarrow \log(n) - M(T) \log\left(\left(1 - \frac{\eta}{2}\right)^{-1}\right) \geq -m_i^{(T)} \log\left((1-\eta)^{-1}\right)$$

$$\Rightarrow M(T) \leq \frac{\log(n) - m_i \log((1-\eta))}{\log\left(\left(1-\frac{\eta}{2}\right)^{-1}\right)}$$

+5

$$\leq \frac{\log(n) + m_i(\eta + \eta^2)}{\log\left(\left(1-\frac{\eta}{2}\right)^{-1}\right)} \quad \text{[Hint]}$$

$$\leq \frac{2\log(n)}{\eta} + m_i(\eta+1)\cdot 2 \quad \left[\log\left((1-x)^{-1}\right) \geq x\right]$$

2. $\frac{5}{3}y^2 + \frac{1}{3}z^2 - y(z+1) \geq 0$

$\Leftrightarrow \quad 5y^2 + z^2 - 3yz - 3y \geq 0 \quad (X)$

$\Leftrightarrow \quad \underbrace{\left(z - \frac{3}{2}y\right)^2}_{A} + \underbrace{\frac{11}{4}y^2 - 3y}_{B} \geq 0$

+5

A is always non-negative.

If $y = 0$, or $y \geq 2$,

B is non-negative.

$\therefore$ need to check $y = 1$.

X becomes:

$2 + z^2 - 3z \geq 0$.

$\Leftrightarrow \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} \geq 0$.

has min at $z = \frac{3}{2}$,

check $z = 1$, $z = 2$. ✓

$\square$

3. $f(S \cap T) + f(S \cup T) \leq f(S) + f(T), \ \forall S, T \quad (1)$

$\Leftrightarrow$

$\forall S \subseteq T, \ x \notin T, \quad (2)$

$f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S)$.

"$\Rightarrow$". Let $S \subseteq T$, $x \notin T$. +3

Let $A = T$

$B = S \cup \{x\}$

From (1),

$f(S) + f(T \cup \{x\}) = f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$

$= f(T) + f(S \cup \{x\})$. ✓

"$\Leftarrow$". $f(T) - f(S \cap T) = \sum_{t_i \in T \setminus X} f(S \cap T \cup \{t_1, \dots, t_i\}) - f(S \cap T \cup \{t_1, \dots, t_{i-1}\})$

+2

$\geq \sum_{t_i \in T \setminus X} f(S \cup \{t_1, \dots, t_i\}) - f(S \cup \{t_1, \dots, t_{i-1}\})$ by (2)

$= f(S \cup T) - f(S)$. $\square$