Homework 2

CSCE631 Intelligent Agents

October 15, 2024

due: October 22, 2024. 11:10am.

May be submitted in class or via email.

1. Imagine the process of picking good times to invest in a stock. For simplicity, assume that there is a single stock of interest, and its daily price movement is modeled as a sequence of binary events: up/down. Each morning, we try to predict whether the price will go up or down that day. If wrong, we lose a dollar, and if correct, we lose nothing.

   Suppose there are a set of n experts who make predictions We apply the "weighted majority algorithm"; let $0 < \eta < 1/2$. Initially, all experts have equal weight; if an expert predicts wrongly on a given day, decrease his weight by a factor of $1 - \eta$. For each day, the algorithm makes the prediction that has a higher total weight of experts making it.

   After T steps, let $m_i^{(T)}$ be the number of mistakes of expert i, and $M^{(T)}$ the number of mistakes our algorithm has made. Then we have the following bound for every i:

   $$M^{(T)} \leq 2(1+\eta)m_i^{(T)} + \frac{2\log n}{\eta}.$$

   In particular, this holds for the best expert. Prove this theorem by proving each step:

   (a) Let $\Phi^{(t)} = \sum_i w_i^{(t)}$, where $w_i^{(t)}$ is the weight of expert i in round t. Show that $\Phi^{(t+1)} \leq \Phi^{(t)}(1 - \eta/2)$.

   (b) Show that $\Phi^{(T+1)} \leq n(1-\eta/2)^{M^{(T)}}$.

   (c) Show the claimed bound by using the relationship $\Phi^{(T+1)} \geq w_i^{(T+1)}$. [Hint: use the fact that $-\log(1-\eta) \leq \eta + \eta^2$]

2. Prove the following statement: For all non-negative integers y, z, $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$.

3. A non-negative function $f : 2^U \to \mathbf{R}$ is submodular iff for all $S, T \subseteq U$,

   $$f(S \cap T) + f(S \cup T) \leq f(S) + f(T).$$

Show this is equivalent to the following property: For all $S \subseteq T$, $x \notin T$,

$$f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S).$$