

CSCE 631 — Sequence Form: Worked Examples

One Kuhn poker game, four representations

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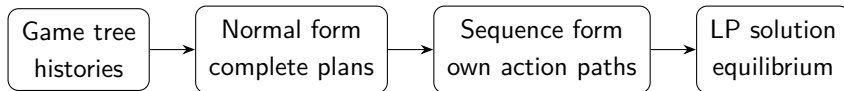
Fall 2026

Texas A&M University

One game, four representations

1. Game tree
2. Normal form
3. Sequence form
4. LP and equilibrium

At every stage, we keep the same information and the same net chip payoffs.



Kuhn poker: rules and payoffs

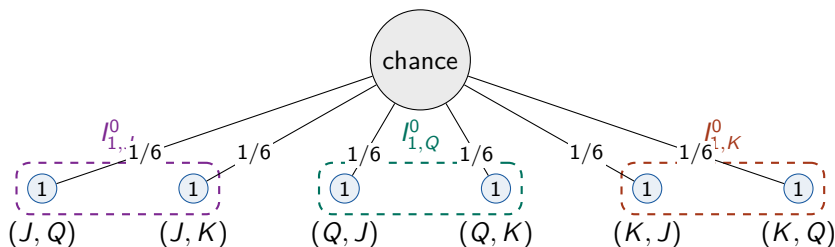
- Three cards: $J < Q < K$. Each player antes 1 chip; deal one private card to each, uniformly without replacement. The third card stays unseen.
- P1 acts first: **check** (c) or **bet 1** (b).
- After c , P2 checks (showdown) or bets 1; after that bet, P1 **calls** (C) or **folds** (F).
- After b , P2 calls or folds. There are no raises. At showdown, the higher card wins.

Payoff means net chips, including the ante

Check–check: winner +1, loser −1. A called bet: winner +2, loser −2. A fold: folder −1, opponent +1. Always $u_2 = -u_1$.

Chance and P1's first information sets

An ordered deal (d, e) means **P1 holds** d , **P2 holds** e . Each of the six deals has probability $1/6$.



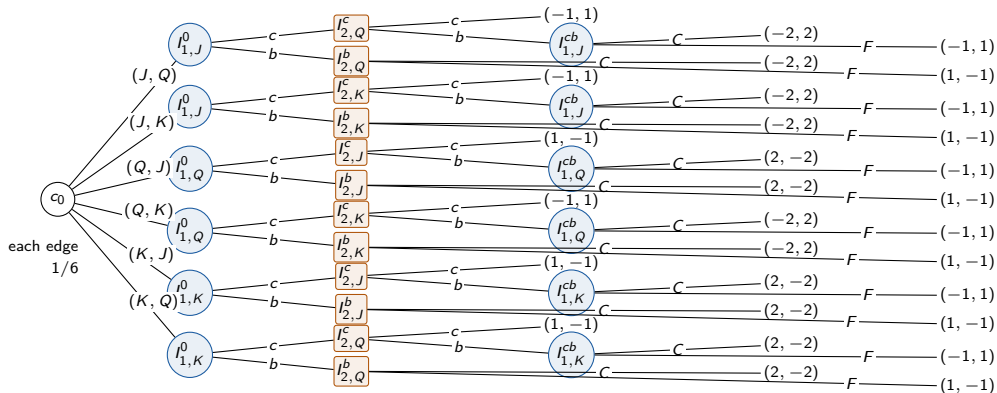
P1 sees its own card, so it can distinguish these three information sets. It cannot distinguish the two nodes inside one dashed box.

Think about it

Can P1 bet differently at (J, Q) and (J, K) ? Why?

The complete tree: all six deals, all 30 terminals

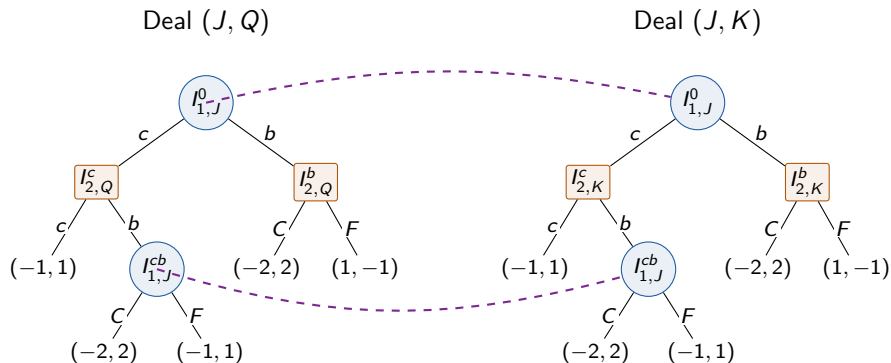
c = check, b = bet, C = call, F = fold. Circles: P1; rectangles: P2. Equal I labels denote the same information set.



1 chance node + 24 decision nodes + 30 leaves = **55 nodes**.

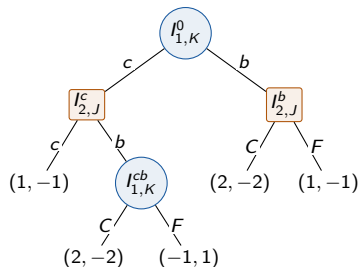
Enlarged branches: P1 holds a Jack

P1 must use the same rule at matching blue nodes in these two branches.



P2's orange nodes are in different information sets: P2 knows whether it holds Q or K.

Trace the chips: deal (K, J) and the payoff ledger



Every deal's five leaves: u_1

Deal	cc	bC	bF	cbC	cbF
JQ	-1	-2	1	-2	-1
JK	-1	-2	1	-2	-1
QJ	1	2	1	2	-1
QK	-1	-2	1	-2	-1
KJ	1	2	1	2	-1
KQ	1	2	1	2	-1

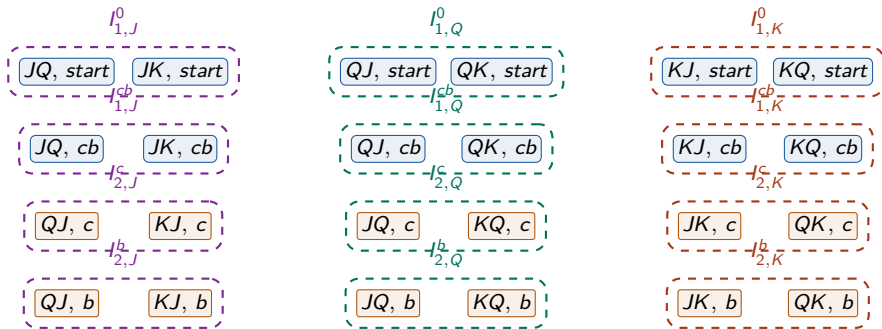
At KJ , bC , the pot is 4. P1 contributed 2 and receives 4: net +2; P2's net is -2.

Think about it

If P2 folds to a bet, why is P1's net gain only 1 chip?

The information-set atlas: group by what the player knows

Each dashed box joins the **two actual nodes** with the listed deal and history.



P1: 3 initial + 3 after $cb = 6$ information sets. **P2:** 3 after c + 3 after $b = 6$.

A pure strategy specifies all six decisions

P1: write $(a_J, a_Q, a_K; r_J, r_Q, r_K)$, where $a_d \in \{c, b\}$ at $I_{1,d}^0$ and $r_d \in \{C, F\}$ at $I_{1,d}^{cb}$.

$$S_1 = \{c, b\}^3 \times \{C, F\}^3, \quad |S_1| = 2^3 \cdot 2^3 = \boxed{64}.$$

P2: write $(a_J, a_Q, a_K; r_J, r_Q, r_K)$, where $a_d \in \{c, b\}$ after P1 checks and $r_d \in \{C, F\}$ after P1 bets.

$$S_2 = \{c, b\}^3 \times \{C, F\}^3, \quad |S_2| = \boxed{64}.$$

For either player, the enumeration is all 8 first triples paired with all 8 second triples:

$$\begin{aligned} &ccc, ccb, cbc, cbb, bcc, bcb, bbc, bbb; \\ &CCC, CCF, CFC, CFF, FCC, FCF, FFC, FFF. \end{aligned}$$

Think about it

Even if P1 chooses *bbb*, why does a full pure strategy still specify its response after *cb*?

A numerical portion of the 64×64 normal form

$M[s_1, s_2]$ averages P1's net payoff over all six deals. Each label lists actions in card order J, Q, K .

	$(ccc; FFF)$	$(ccc; CCC)$	$(bbb; FFF)$	$(bbb; CCC)$
$(ccc; FFF)$	0	0	-1	-1
$(ccc; CCC)$	0	0	0	0
$(bbb; FFF)$	1	0	1	0
$(bcb; FCC)$	2/3	0	2/3	0

For example, $(ccc; FFF)$ against $(bbb; FFF)$ always reaches cbF : $M = \frac{1}{6} \sum_{\text{six deals}} (-1) = -1$.

Normal-form LP for P1

$\max_{p,v} v$ subject to $M^\top p \geq v\mathbf{1}$, $\mathbf{1}^\top p = 1$, $p \geq 0$. This is what we'd have to solve with a standard LP if we didn't have the sequence form.

Here: $64^2 = 4,096$ matrix entries versus 55 tree nodes. In general, binary information sets give $2^{|I_i|}$ pure strategies: the blowup is exponential in the number of information sets.

List the sequences: keep the information-set labels

As in L12, q_i is an ordered list of *that player's own* (I, a) pairs. Chance and the opponent's actions are excluded; their observations identify I . **Think about it:** Are “check with J” and “check with Q” the same sequence?

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For each $d \in \{J, Q, K\}$, use these abbreviations:

P1		P2	
$c_d = ((I_{1,d}^0, c))$	check	$h_d = ((I_{2,d}^c, c))$	check after c
$b_d = ((I_{1,d}^0, b))$	bet	$t_d = ((I_{2,d}^c, b))$	bet after c
$r_d = c_d \cdot (I_{1,d}^{cb}, C)$	check, then call	$a_d = ((I_{2,d}^b, C))$	call after b
$s_d = c_d \cdot (I_{1,d}^{cb}, F)$	check, then fold	$f_d = ((I_{2,d}^b, F))$	fold after b

The **fixed row/column orders** for all following matrices are

$$Q_1 = (\emptyset, c_J, c_Q, c_K, b_J, b_Q, b_K, r_J, r_Q, r_K, s_J, s_Q, s_K),$$

$$Q_2 = (\emptyset, h_J, h_Q, h_K, t_J, t_Q, t_K, a_J, a_Q, a_K, f_J, f_Q, f_K).$$

13 sequences each, not five unlabelled action patterns. Every nonempty P2 sequence contains just one P2 action.

The two flow matrices, row by row

E_1 (Player 1)

	$\emptyset$	c_J	c_Q	c_K	b_J	b_Q	b_K	r_J	r_Q	r_K	s_J	s_Q	s_K
0	1	0	0	0	0	0	0	0	0	0	0	0	0
$I_{1,J}^0$	-1	1	0	0	1	0	0	0	0	0	0	0	0
$I_{1,Q}^0$	-1	0	1	0	0	1	0	0	0	0	0	0	0
$I_{1,K}^0$	-1	0	0	1	0	0	1	0	0	0	0	0	0
$I_{1,J}^{cb}$	0	-1	0	0	0	0	0	1	0	0	1	0	0
$I_{1,Q}^{cb}$	0	0	-1	0	0	0	0	0	1	0	0	1	0
$I_{1,K}^{cb}$	0	0	0	-1	0	0	0	0	0	1	0	0	1

E_2 (Player 2)

	$\emptyset$	h_J	h_Q	h_K	t_J	t_Q	t_K	a_J	a_Q	a_K	f_J	f_Q	f_K
0	1	0	0	0	0	0	0	0	0	0	0	0	0
$I_{2,J}^c$	-1	1	0	0	1	0	0	0	0	0	0	0	0
$I_{2,Q}^c$	-1	0	1	0	0	1	0	0	0	0	0	0	0
$I_{2,K}^c$	-1	0	0	1	0	0	1	0	0	0	0	0	0
$I_{2,J}^b$	-1	0	0	0	0	0	0	1	0	0	1	0	0
$I_{2,Q}^b$	-1	0	0	0	0	0	0	0	1	0	0	1	0
$I_{2,K}^b$	-1	0	0	0	0	0	0	0	0	1	0	0	1

For every $d \in \{J, Q, K\}$, the two kinds of information-set rows say

$$\begin{aligned} x_1(c_d) + x_1(b_d) &= x_1(\emptyset), & x_1(r_d) + x_1(s_d) &= x_1(c_d), \\ x_2(h_d) + x_2(t_d) &= x_2(\emptyset), & x_2(a_d) + x_2(f_d) &= x_2(\emptyset). \end{aligned}$$

Both E_i are 7×13 ; $E_i x_i = e_i$, $e_i = (1, 0, 0, 0, 0, 0, 0)^\top$. Row 0 sets $x_i(\emptyset) = 1$; other rows have -1 at the parent, $+1$ at children.

Weights exclude chance: each card has a separate flow equation.

Build A from terminal nodes: five hand calculations

$$A[q_1, q_2] = \sum_{\substack{z: q_1(z)=q_1 \\ q_2(z)=q_2}} u_1(z) \pi_c(z), \quad \pi_c(z) = \frac{1}{6}.$$

Entry	Matching deal and public history	Contribution
$A[b_K, a_J]$	(K, J) , P1 bets, P2 calls	$(+2)(1/6) = 1/3$
$A[c_J, h_Q]$	(J, Q) , check-check	$(-1)(1/6) = -1/6$
$A[r_Q, t_K]$	(Q, K) , check-bet-call	$(-2)(1/6) = -1/3$
$A[s_Q, t_J]$	(Q, J) , check-bet-fold	$(-1)(1/6) = -1/6$
$A[b_J, f_K]$	(J, K) , bet-fold	$(+1)(1/6) = 1/6$

Each of these pairs matches **exactly one terminal**; its sum has one term. For example, $A[b_K, a_K] = 0$ because the same card cannot be dealt twice. **Think about it:** Why is $A[c_Q, t_K] = 0$, although check-bet is possible?

Answer: it is not terminal. P1 must continue to r_Q or s_Q . Realization weights exclude chance; the factor $1/6$ enters A exactly once.

The full sequence-form payoff matrix

Rows are $\mathcal{Q}_1$, columns are $\mathcal{Q}_2$. **The table is 6A:** multiply every entry by 1/6 to obtain A.

	$\emptyset$	h_J	h_Q	h_K	t_J	t_Q	t_K	a_J	a_Q	a_K	f_J	f_Q	f_K
$\emptyset$	0	0	0	0	0	0	0	0	0	0	0	0	0
c_J	0	0	-1	-1	0	0	0	0	0	0	0	0	0
c_Q	0	1	0	-1	0	0	0	0	0	0	0	0	0
c_K	0	1	1	0	0	0	0	0	0	0	0	0	0
b_J	0	0	0	0	0	0	0	0	-2	-2	0	1	1
b_Q	0	0	0	0	0	0	0	2	0	-2	1	0	1
b_K	0	0	0	0	0	0	0	2	2	0	1	1	0
r_J	0	0	0	0	0	-2	-2	0	0	0	0	0	0
r_Q	0	0	0	0	2	0	-2	0	0	0	0	0	0
r_K	0	0	0	0	2	2	0	0	0	0	0	0	0
s_J	0	0	0	0	0	-1	-1	0	0	0	0	0	0
s_Q	0	0	0	0	-1	0	-1	0	0	0	0	0	0
s_K	0	0	0	0	-1	-1	0	0	0	0	0	0	0

$U_1 = x_1^T A x_2$. A: $13 \times 13 = 169$ entries, **30 nonzeros**; normal form: $64 \times 64 = 4,096$.

P1's LP: L12 with all coefficients filled in

L12: $\max e_2^\top v_2$ subject to $E_1 x_1 = e_1$, $A^\top x_1 - E_2^\top v_2 \geq 0$, $x_1 \geq 0$. Write $v_2 = (v_0, u_J, u_Q, u_K, w_J, w_Q, w_K)$ and abbreviate $c_d = x_1(c_d)$, $b_d = x_1(b_d)$, $r_d = x_1(r_d)$, $s_d = x_1(s_d)$.

$$\begin{aligned} \max \quad & v_0 \\ \text{s.t.} \quad & x_1(\emptyset) = 1, \quad c_d + b_d = 1, \quad r_d + s_d = c_d \quad (d = J, Q, K), \\ & v_0 \leq u_J + u_Q + u_K + w_J + w_Q + w_K. \end{aligned}$$

The remaining 12 inequalities are the table entries (one per nonempty P2 sequence):

columns of A	J	Q	K
h_d	$6u_J \leq c_Q + c_K$	$6u_Q \leq -c_J + c_K$	$6u_K \leq -c_J - c_Q$
t_d	$6u_J \leq 2r_Q + 2r_K - s_Q - s_K$	$6u_Q \leq -2r_J + 2r_K - s_J - s_K$	$6u_K \leq -2r_J - 2r_Q - s_J - s_Q$
a_d	$6w_J \leq 2b_Q + 2b_K$	$6w_Q \leq -2b_J + 2b_K$	$6w_K \leq -2b_J - 2b_Q$
f_d	$6w_J \leq b_Q + b_K$	$6w_Q \leq b_J + b_K$	$6w_K \leq b_J + b_Q$

$x_1 \geq 0$; all seven v_2 variables are unrestricted in sign.

Size: 20 variables, 7 equalities, 13 payoff inequalities, 13 nonnegativity bounds.

One optimal pair of realization plans

Use the sequence orders from Stage 3. Both empty-sequence weights equal 1.

Player 1: x_1^*				Player 2: x_2^*			
Sequence	J	Q	K	Sequence	J	Q	K
c_d	2/3	1	0	h_d	2/3	1	0
b_d	1/3	0	1	t_d	1/3	0	1
r_d	0	2/3	0	a_d	0	1/3	1
s_d	2/3	1/3	0	f_d	1	2/3	0

For example, the P1 flow at $I_{1,J}^{cb}$ is $0 + 2/3 = 2/3$; at $I_{1,Q}^{cb}$ it is $2/3 + 1/3 = 1$. Thus $E_i x_i^* = e_i$.

Expected payoff, in deal order JQ, JK, QJ, QK, KJ, KQ :

$$x_1^{*\top} A x_2^* = \frac{1}{6} \left(-\frac{2}{3} - \frac{4}{3} + 1 - \frac{5}{3} + 1 + \frac{4}{3} \right) = \boxed{-\frac{1}{18}}.$$

These are own-action products, not probabilities of being dealt each card.

Recover the complete behavioral equilibrium

For $x_i(q(I)) > 0$, L12 gives $\beta_i(a \mid I) = \frac{x_i(q(I) \cdot (I, a))}{x_i(q(I))}$.

Action probability	J	Q	K
P1: bet initially	1/3	0	1
P1: call after check–bet	0	2/3	1 [†]
P2: bet after P1 checks	1/3	0	1
P2: call after P1 bets	0	1/3	1

The other action has the complementary probability. For instance, $\beta_1(C \mid I_{1,Q}^{cb}) = (2/3)/1 = 2/3$ and $\beta_1(F \mid I_{1,J}^{cb}) = (2/3)/(2/3) = 1$.

† P1 never checks K: $x_1(c_K) = 0$, so the ratio is 0/0. Choose call with probability 1 there; this off-path choice leaves x_1^* unchanged.

The familiar endpoint equilibrium: P1 bets K, checks Q, bluffs J at 1/3; P2 calls K, folds J, and calls Q at 1/3.

Reference: Hoehn et al. (2005, AAAI), *Opponent Exploitation in Simplified Poker*, p. 2.

Verify optimality: matching primal and dual certificates

Let $X_i = \{x_i \geq 0 : E_i x_i = e_i\}$. An expected payoff alone does not prove equilibrium. Use L12's dual:
 $\min e_1^\top v_1$, with $E_2 x_2 = e_2$, $E_1^\top v_1 - A x_2 \geq 0$, $x_2 \geq 0$.

$$v_2^* = \left(-\frac{1}{18}, \frac{1}{6}, -\frac{1}{9}, -\frac{7}{18}, \frac{1}{6}, \frac{2}{9}, -\frac{1}{9}\right)^\top,$$
$$v_1^* = \left(-\frac{1}{18}, -\frac{1}{3}, -\frac{1}{9}, \frac{7}{18}, -\frac{1}{6}, -\frac{2}{9}, \frac{1}{9}\right)^\top.$$

The coordinates follow the row orders of E_2 and E_1 , respectively. Direct multiplication gives

$$A^\top x_1^* - E_2^\top v_2^* = \left(0; 0, 0, \frac{1}{9}; 0, 0, 0; \frac{1}{6}, 0, 0; 0, 0, \frac{1}{6}\right)^\top \geq 0,$$
$$E_1^\top v_1^* - A x_2^* = \left(0; 0, 0, 0; 0, \frac{1}{18}, 0; \frac{1}{6}, 0, 0; 0, 0, \frac{1}{6}\right)^\top \geq 0.$$

Both objective values equal $-1/18$, so weak duality certifies optimality:

$$\forall x_1 \in X_1, x_2 \in X_2 : \quad x_1^\top A x_2^* \leq -\frac{1}{18} \leq x_1^{*\top} A x_2.$$

Independent check: enumeration of all 64 pure deviations per player gives

$$\max_{s_1} U_1(s_1, x_2^*) = \min_{s_2} U_1(x_1^*, s_2) = -1/18.$$

Four representations of the same game

Representation	What is stored	Size for this game
Game tree	Histories, information sets, chance, leaf payoffs	55 nodes; 30 leaves; 6 information sets per player
Normal form	Payoff for each pair of complete pure plans	$64 \times 64 = 4,096$ entries
Sequence form	Own action sequences, flow matrices, sparse payoffs	13 sequences per player; $E_i : 7 \times 13$; $A : 13 \times 13$, 30 nonzeros
P1's sequence LP	One plan and the opponent's dual flow values	20 variables; 7 equalities; 13 payoff inequalities

What the sequence form saves

Perfect recall lets flow constraints encode strategic possibilities without enumerating exponentially many complete plans. The equilibrium value remains $-1/18$ for P1.